dune-dpg: A Library for Discontinous Petrov Galerkin Using Optimal Test Spaces

Felix Gruber and Angela Klewinghaus

joint work with Olga Mula

IGPM, RWTH Aachen

Dune User Meeting 2015 in Heidelberg

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Motivation:

Variational formulation:

Find $u \in \mathcal{U}$ such that

$$b(u, v) = \ell(v) \qquad \text{for all } v \in \mathcal{V}$$

for Hilbert spaces $\mathcal{U}, \mathcal{V}$, continuous bilinear form $b : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ and $\ell \in \mathcal{V}'$.

Goal: Stability

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- Discrete approximation yields a best approximation (up to a constant)

$$\|u - u_h\|_{\mathcal{U}} \lesssim \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}} \,.$$

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- Residual-based error bound

$$\|u - u_h\|_{\mathcal{U}} \simeq \|b(u_h, \cdot) - \ell\|_{\mathcal{V}'} \ .$$

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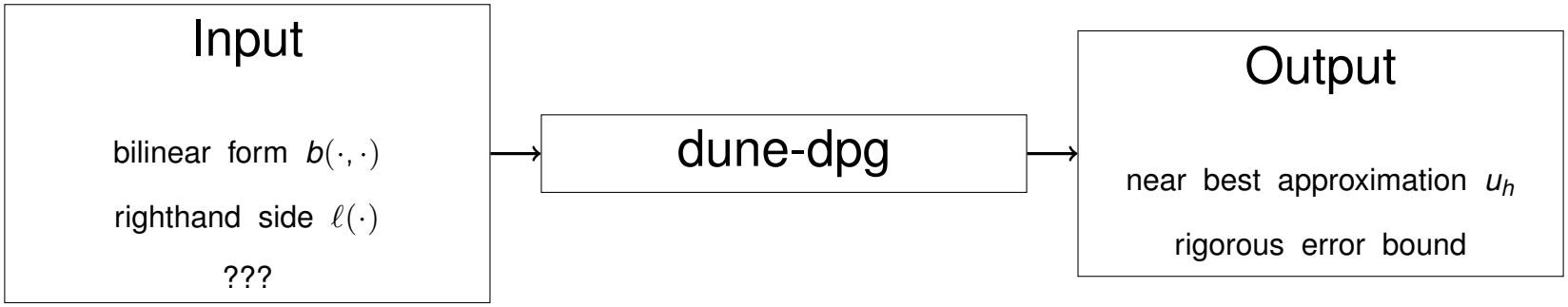
Ultimate Goal:



Choose $\mathcal{U}_h$ to Ensure Good Approximation Properties, Choose $\mathcal{V}_h$ to Ensure Stability

How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

Literature: Barret/Morton, Demkowicz/Gopalakrishnan, Dahmen/Cohen/Welper, Dahmen/Huang/Schwab/Welper, Broersen/Stevenson, Broersen/Dahmen/Stevenson

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How to choose $\mathcal{V}_h$ for given $\mathcal{U}_h$?

- Optimal test norm

$$\|\cdot\|_{\mathcal{V},opt} := \sup_{u\in\mathcal{U}} \frac{b(u,v)}{\|u\|_{\mathcal{U}}}$$

  - yields residual based error bound

$$\|u_h - u\|_{\mathcal{U}} = ||b(u_h,\cdot) - \ell||_{\mathcal{V}',opt}$$

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- Optimal test space $\mathcal{V}_h = B^{-*}R_{\mathcal{U}}\mathcal{U}_h$, where $R_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}'$ is the Rieszisomorphism, that is $\mathcal{V}_h := T(\mathcal{U}_h)$ where

$$\langle T(u_h), v\rangle_{\mathcal{V},opt} = b(u_h, v)\,, \qquad \forall\, v \in \mathcal{V}$$

  - yields best approximation property

$$\|u - u_h\|_{\mathcal{U}} = \inf_{w_h\in\mathcal{U}_h} \|u - w_h\|_{\mathcal{U}}$$

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  - use an ultraweak variational formulation
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We can compute an approximation for $\mathcal{V}_h$ by solving

$$\langle \hat{T}(u_h), v\rangle_{\mathcal{V},K} = b_K(u_h, v)\,, \qquad \forall\, v \in \hat{\mathcal{V}}\big|_K\,,$$

where K is a single cell and $\langle\cdot,\cdot\rangle_{\mathcal{V},K}$ and $b_K(\cdot,\cdot)$ denote restrictions to K.

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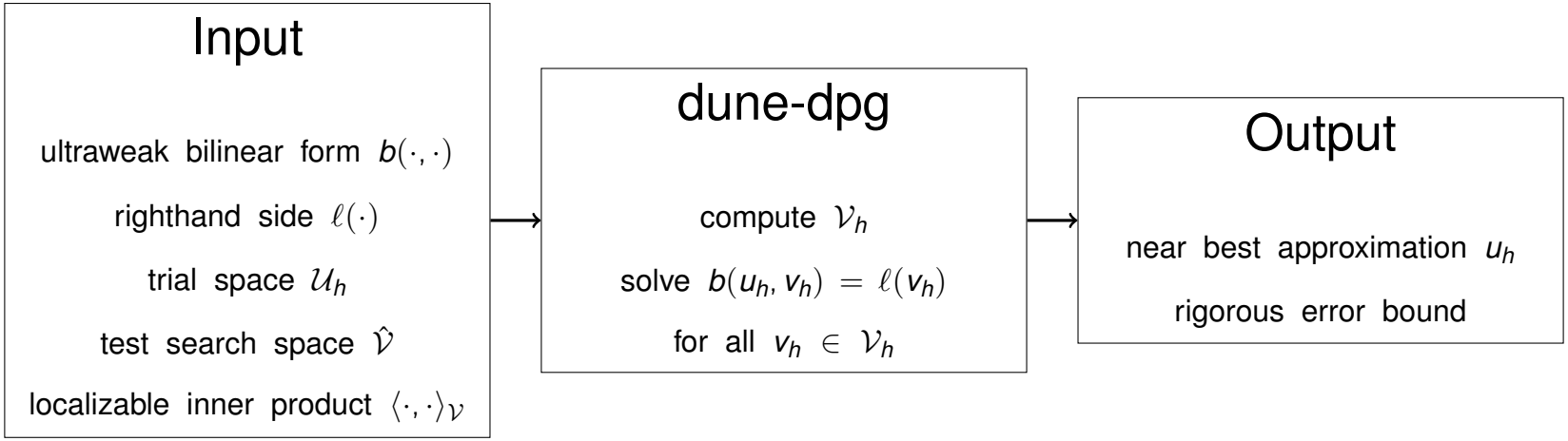
dune-dpg:



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Implementation: Layout of dune-dpg

rough overview of a DPG solver:

```
define grid
define trial space U_h and test search space V̂
define bilinear form and inner product

assemble system matrix and rhs vector
interpolate boundary values

solve system
compute a posteriori error estimates
[refine adaptively]
```

Implementation: BilinearForm / InnerProduct

- In our case, a bilinear form can be seen as a sum of integral terms, e. g.

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- Abstractly, an inner product is simply a symmetric bilinear form over a single space. Thus, we can reuse the "sum over `IntegralTerms`" logic to define an `InnerProduct` class.

Implementation: Optimal Testspace

- Localfunctions: Linear combinations of localfunctions of the test search space
  - Parameter: Localfunctions Test Search Space, Coefficient Matrix
- Functions:
  - Parameter: Trial Spaces, Test Search Spaces, Bilinear Form, Inner Product
  - uses IndexSet of Trial Spaces
  - in LocalView.bind(): Uses Trial Spaces, Test Search Spaces, Bilinear Form, Inner Product to set up and solve

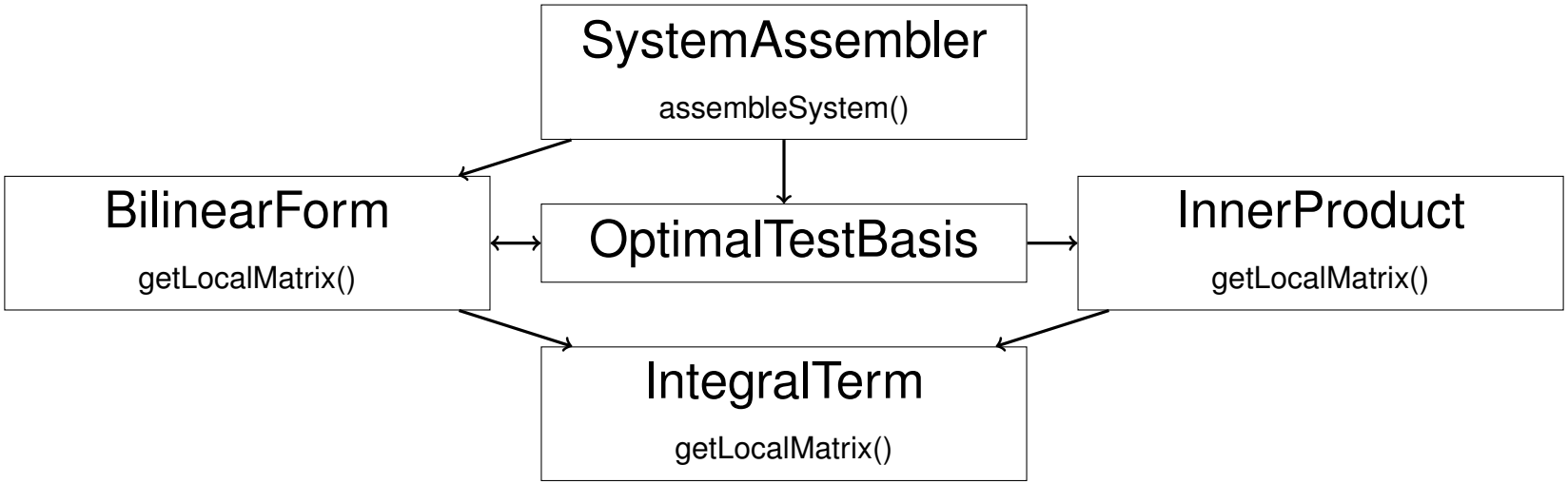
$$\langle \hat{T}(u_h), v \rangle_{\mathcal{V},K} = b_K(u_h, v)\,, \qquad \forall\, v \in \hat{\mathcal{V}}\big|_K\,,$$

  for every (local) solution basis function u_h on the current (physical) cell.
  $\rightarrow$ yields coefficient matrix for Localfunctions

Implementation: SystemAssembler

- Building a global system matrix and right hand side for our method is handled by the `SystemAssembler` class.
- The `SystemAssembler` loops over all cells and uses the `BilinearForm` and `OptimalTestBasis` to build local matrices.
- The local matrices built by the `BilinearForm` are then copied into a global system matrix. (Thanks to dune-functions we have a mapping between local and global indices.)
- Similarily the right hand side of the system is assembled cell by cell.
- `SystemAssembler` also later applies the boundary values to the system.

Implementation: Overview of dune-dpg



Example: Transport

Transport equation:

$$\begin{aligned} \beta \cdot \nabla u + cu &= f \quad \text{on } \Omega, \\ u &= g \quad \text{on } \Gamma_-. \end{aligned}$$

Challenge:

- not "second order elliptic" but hyperbolic

Theoreticel foundations:

- D. Broersen, W. Dahmen, R. Stevenson

Applications:

- discrete ordinates discretization of Boltzman equation (F. Gruber, joint work with W. Dahmen, O. Mula)
- limit case of the convection-diffusion equation (A. Klewinghaus, joint work with W. Dahmen)
- porous media (W. Dahmen, V. König, S. Müller)

We'd like to thank Oliver Sander for his introduction into dune(-functions).

Results Transport

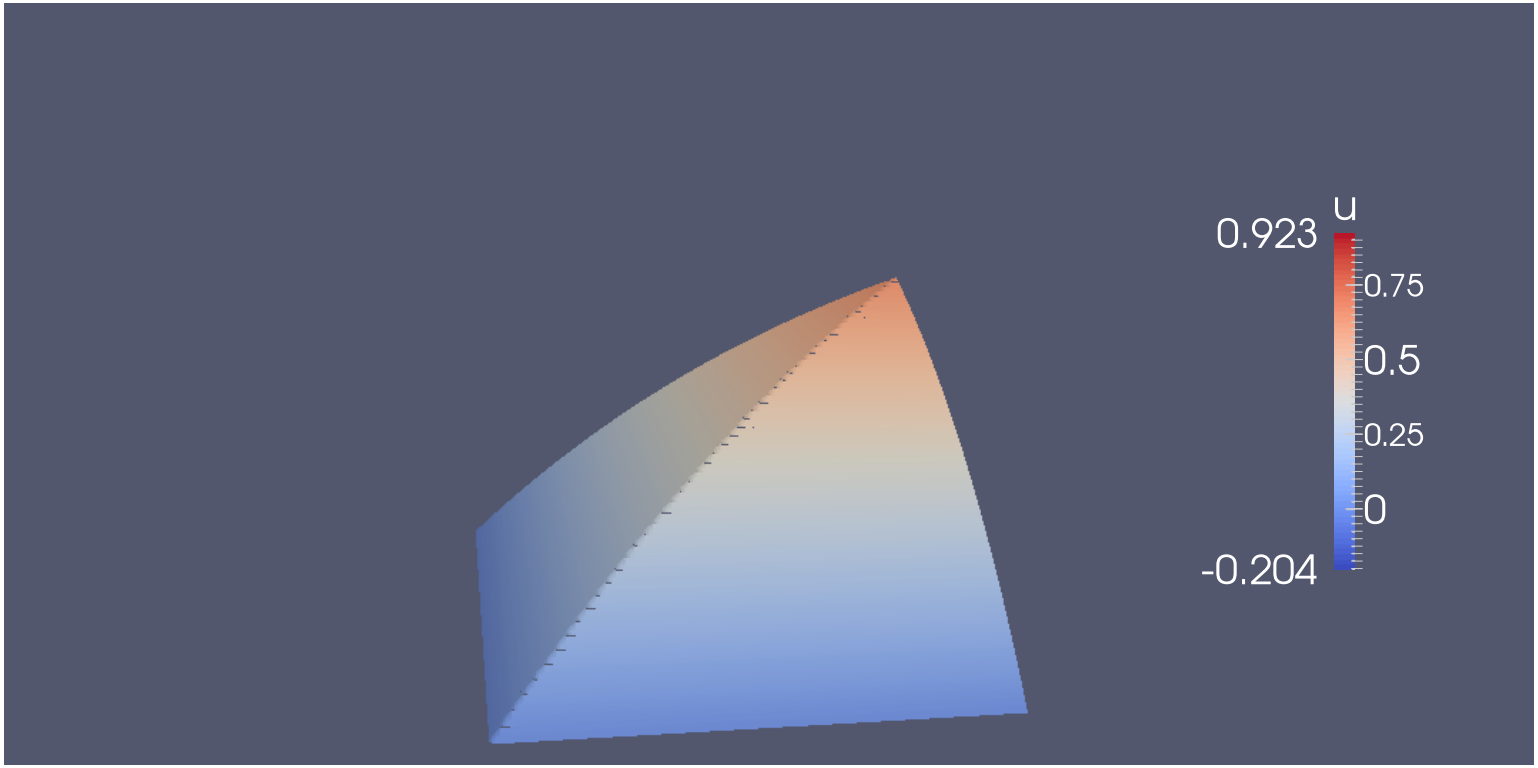


Figure: $\beta = (1, 1)$, $c = 1$, $f = 1$

Results Transport

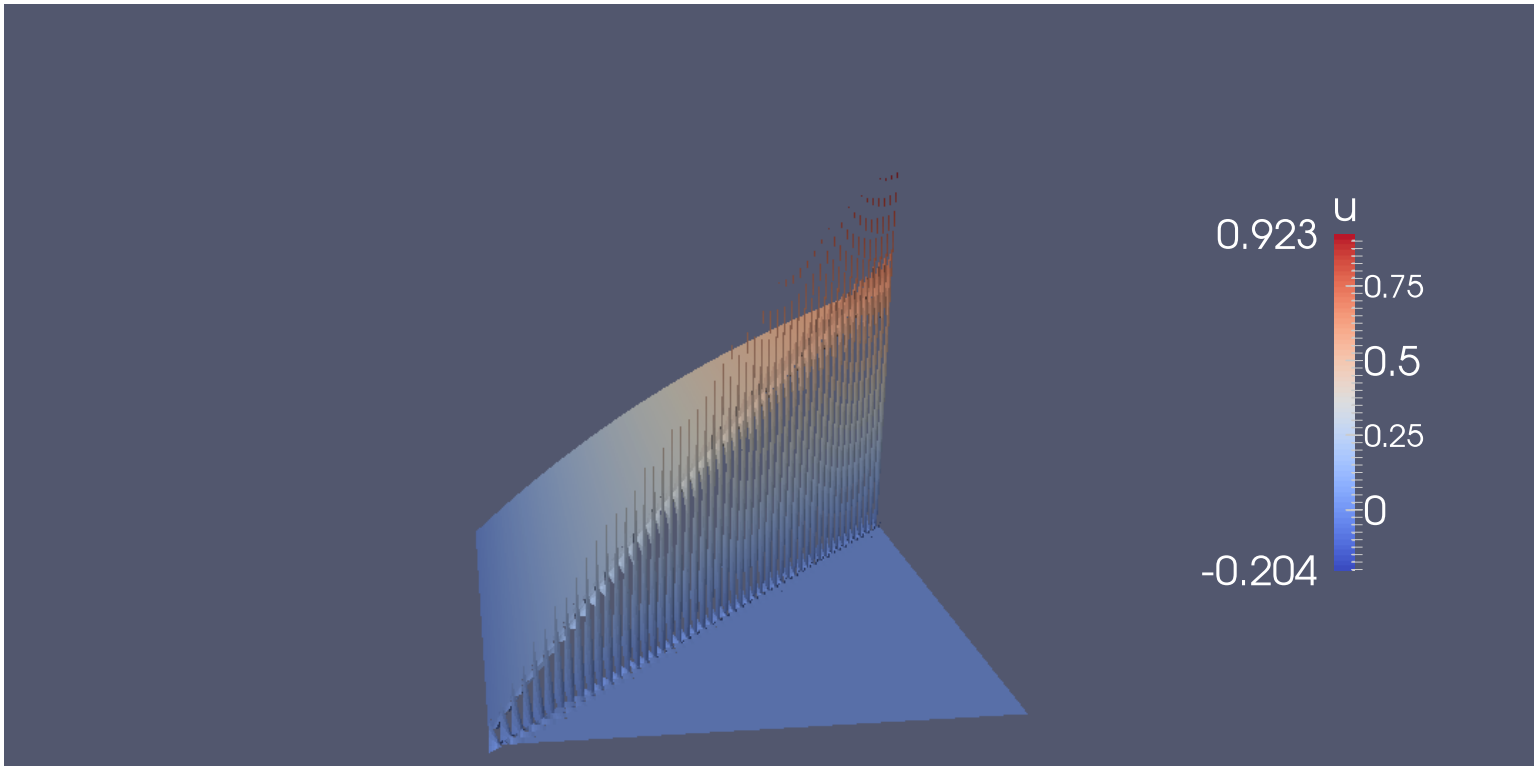
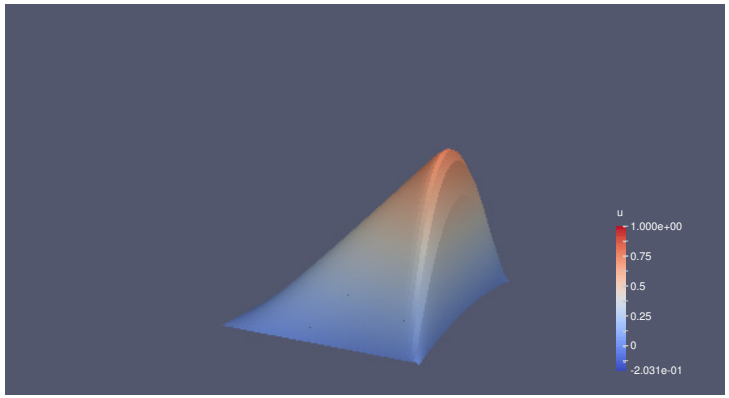
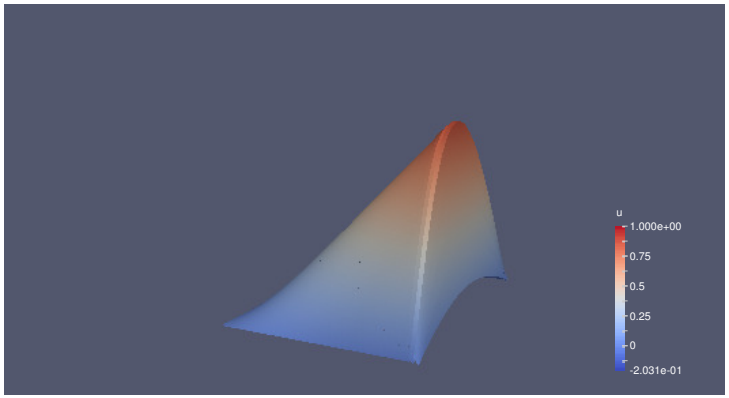


Figure: $\beta = (1, 1)$, $c = 1$, $f = 1$, if $x_1 < x_2$, or $f = 0$ else

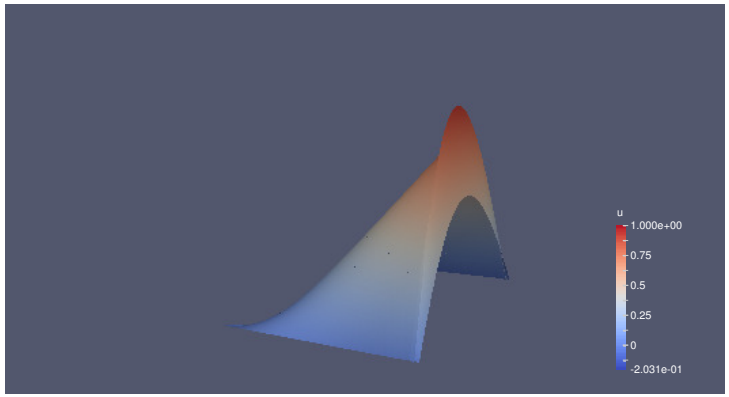
Results Convection Diffusion



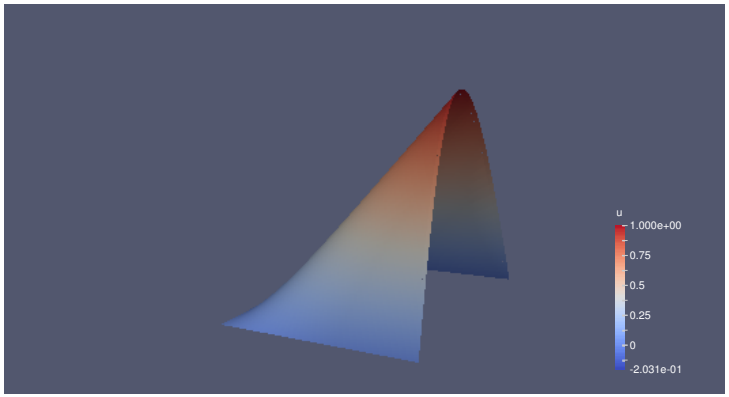
(a) $\epsilon = 0.02$



(b) $\epsilon = 0.01$



(c) $\epsilon = 0.002$



(d) $\epsilon = 0.00001$

Figure: $\Omega = (0,1)^2$, regular square grid with 50×50 elements ($h = 0.02$)